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TORSIONAL OSCILLATION OF A NON-HOMOGENEOUS ANISOTROPIC TUBE OF FINITE LENGTH

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Abstract—This paper treats the torsional vibrations in a finite circular tube of an anisotropic inhomogeneous material. The general solution for the problem is obtained. Both free and forced torsional vibrations of tubes can be treated as special cases of the problem. The values of the natural frequencies including the higher modes are computed for such geometrical parameters as the ratio of the radii of a tube and the length. Also, the values of natural frequencies in terms of the various values of the parameter of anisotropy are calculated. It is interesting to note that the presence of the central hole in a tube has no effect upon the natural frequencies of the rigid mode, but for the case of nodal modes the natural frequencies change with the thickness of the tube.

INTRODUCTION

THE torsional vibration of an isotropic circular cylinder has been extensively investigated. A solution of elastic isotropic circular bars subjected to torsional vibration was obtained by Love [1]. Torsional waves in elastic isotropic media have been considered by Kolsky [2] and Clark [3]. Torsional oscillations in anisotropic media were discussed by Chakravorty [4]. Recently, Wainwright [5] derived a more general solution for isotropic cylinders of finite length. Stanisic and Osborn [6] treated the torsional vibration of an inhomogeneous anisotropic hollow cylinder. (However, the authors of paper [6] used the stress–strain laws for isotropic media instead of using the generalized Hooke's laws for anisotropic media, according to equation (1) of their paper; thus the results obtained cannot apply to anisotropic media.) Paul [7] investigated the torsional vibration of a circular cylinder of piezoelectric β-Quartz.

Many modern design components involve finite hollow cylinders with new materials such as fiber-reinforced composite materials. Since these materials are essentially elastic and anisotropic, and even inhomogeneous, the character of the anisotropy and of the inhomogeneity must be taken into consideration in the stress analysis. Therefore, the torsional vibration of an inhomogeneous, anisotropic circular tube of finite length will be investigated in this paper. The result for an isotropic homogeneous solid shaft can be obtained as a special case of the study. Moreover, it was noted by Kolsky [2] that some complex modes of torsional vibration (other than the modes observed by Love [1], for which each transverse section of a cylinder rotates as a whole) are involved in the motion. For simplicity, in this paper the former modes will be called modal modes and the latter ones will be called rigid modes. Here, the authors seek the determination of a more general solution for the torsional vibration problem including all higher modes. The basic equations of the problem are derived from the general theory of elasticity, so that both free and forced torsional vibration of the tube can be treated as special cases of the study. Determination of the general solution for forced vibration requires the specification of either the appropriate stress distribution or the possible displacement function at the two ends of the tube. The effects of anisotropy and inhomogeneity of the material of the tube on the torsional motion will be illustrated.

BASIC EQUATIONS

A cylindrically orthotropic inhomogeneous circular tube of finite length is considered. The r, θ and z of cylindrical coordinates are chosen as the reference frame. The equations of motion are

$$\begin{aligned}
\sigma_{r,r}+\frac{1}{r}\tau_{r\theta,\theta}+\tau_{rz,z}+\frac{1}{r}(\sigma_r-\sigma_\theta) &= \rho\frac{\partial^2 u}{\partial t^2},\\
\tau_{r\theta,r}+\frac{1}{r}\sigma_{\theta,\theta}+\tau_{\theta z,z}+\frac{2}{r}\tau_{r\theta} &= \rho\frac{\partial^2 v}{\partial t^2},\\
\tau_{rz,r}+\frac{1}{r}\tau_{\theta z,\theta}+\sigma_{z,z}+\frac{1}{r}\tau_{rz} &= \rho\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{1}$$

where $\{\sigma_i\}$ are normal stress components, $\{\tau_{ij}\}$ are shearing stress components, $\{u, v, w\}$ are the components of the displacement vector in the directions r, θ and z respectively, ρ is the density, and t is time.

The stress–strain laws for a cylindrically orthotropic medium are

$$\begin{aligned}
\sigma_r &= C_{11}\varepsilon_r+C_{12}\varepsilon_\theta+C_{13}\varepsilon_z,\\
\sigma_\theta &= C_{12}\varepsilon_r+C_{22}\varepsilon_\theta+C_{23}\varepsilon_z,\\
\sigma_z &= C_{13}\varepsilon_r+C_{23}\varepsilon_\theta+C_{33}\varepsilon_z,\\
\tau_{\theta z} &= C_{44}\gamma_{\theta z},\\
\tau_{rz} &= C_{55}\gamma_{rz},\\
\tau_{r\theta} &= C_{66}\gamma_r
\end{aligned} \tag{2}$$

where the $\{C_{ij}\}$ are elastic moduli which are functions of the space coordinates, the $\{\varepsilon_i\}$ denote the components of normal strain, and the $\{\gamma_{ij}\}$ are components of shearing strain.

The strain–displacement relations in the cylindrical polar coordinate system are

$$\begin{aligned}
&\varepsilon_r = u_{,r}, \qquad \varepsilon_\theta = \frac{1}{r}(v_{,\theta}+u), \qquad \varepsilon_z = w_{,z},\\
&\gamma_{\theta z} = v_{,z}+\frac{1}{r}w_{,\theta},\\
&\gamma_{rz} = w_{,r}+u_{,z},\\
&\gamma_{r\theta} = \frac{1}{r}u_{,\theta}+v_{,r}-\frac{v}{r}.
\end{aligned} \tag{3}$$

In the case of torsional vibration of a circular tube, the components of the displacement vector can be assumed as

$$u = 0, \qquad w = 0, \qquad v = v(r, z, t). \tag{4}$$

By equation (4), the components of strain (3) reduce to

$$\begin{aligned} \varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{rz} = 0, \\ \gamma_{\theta z} = \frac{\partial v}{\partial z}, \\ \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \end{aligned} \tag{5}$$

and the components of stress (2) reduce to

$$\begin{aligned} \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \\ \tau_{\theta z} = C_{44}\frac{\partial v}{\partial z}, \\ \tau_{r\theta} = C_{66}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right). \end{aligned} \tag{6}$$

If the values of stresses from (6) are substituted in equation (1), the first and third equations of (1) are satisfied identically, while the second equation of (1) yields

$$\frac{\partial C_{66}}{\partial r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + C_{66}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \frac{\partial C_{44}}{\partial z}\frac{\partial v}{\partial z} + C_{44}\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{7}$$

For simplicity, it is assumed throughout this paper that the material moduli follow the same distribution law, such as

$$\begin{aligned} C_{66} = \tilde{C}_{66}\,\mathrm{e}^{-kz}, \\ C_{44} = \tilde{C}_{44}\,\mathrm{e}^{-kz}, \\ \rho = \tilde{\rho}\,\mathrm{e}^{-kz}, \end{aligned} \tag{8}$$

where $\tilde{C}_{66}$, $\tilde{C}_{44}$, $\tilde{\rho}$ and k are constants.

The substitution of equation (8) in equation (7) yields

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\tilde{C}_{44}}{\tilde{C}_{66}}\left(\frac{\partial^2 v}{\partial z^2} - k\frac{\partial v}{\partial z}\right) = \frac{\tilde{\rho}}{\tilde{C}_{66}}\frac{\partial^2 v}{\partial t^2}. \tag{9}$$

SOLUTION OF THE PROBLEM

The solution of equation (9) is assumed to be of the form

$$v(r, z, t) = R(z)Z(z)\,\mathrm{e}^{i\omega t}, \tag{10}$$

where ω denotes angular frequency, and $i = \sqrt{-1}$.

Then equation (9) becomes

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\frac{\tilde{\rho}}{\tilde{C}_{66}}\omega^2-\frac{1}{r^2}\right)R\right) = -\frac{\tilde{C}_{44}}{\tilde{C}_{66}}\frac{1}{Z}\left(\frac{d^2Z}{dz^2}-k\frac{dZ}{dz}\right) = -p^2 \tag{11}$$

where p^2 is the separation constant. In turn, equation (11) yields

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\alpha^2-\frac{1}{r^2}\right)R = 0, \tag{12}$$

$$\frac{d^2Z}{dz^2}-k\frac{dZ}{dz}-\gamma^2 Z = 0, \tag{13}$$

where

$$\alpha^2 = \beta^2+p^2, \qquad \beta^2 = \frac{\tilde{\rho}}{\tilde{C}_{44}}\omega^2, \quad \text{and} \quad \gamma^2 = \frac{\tilde{\rho}}{\tilde{C}_{44}}p^2. \tag{14}$$

The solution of equation (12) is

$$R(r) = AJ_1(\alpha r)+BY_1(\alpha r), \tag{15}$$

and the solution of equation (13) is

$$Z(z) = C\,e^{\lambda' z}+D\,e^{\lambda'' z}, \tag{16}$$

where

$$\lambda' = \tfrac{1}{2}[k+\sqrt{(k^2+4\gamma^2)}], \quad \text{and} \quad \lambda'' = \tfrac{1}{2}[k-\sqrt{(k^2+4\gamma^2)}], \tag{17}$$

and A, B, C and D are integration constants.

Thus, the displacement can be expressed as

$$v_1(r,z,t) = \{AJ_1(\alpha r)+BY_1(\alpha r)\}\{C\,e^{\lambda' z}+D\,e^{\lambda'' z}\}\,e^{i\omega t}. \tag{18}$$

Now, from equation (6), the expressions for the nonvanishing stress components are obtained

$$\tau'_{\theta z} = \tilde{C}_{44}\{AJ_1(\alpha r)+BY_1(\alpha r)\}\{C\lambda'\,e^{(\lambda'-k)z}+D\lambda''\,e^{(\lambda''-k)z}\}\,e^{i\omega t}, \tag{19}$$

$$\tau'_{r\theta} = -\tilde{C}_{66}\{A\alpha J_2(\alpha r)+B\alpha Y_2(\alpha r)\}\{C\,e^{(\lambda'-k)z}+D\,e^{(\lambda''-k)z}\}\,e^{i\omega t}. \tag{20}$$

It is noted that the solutions given in equations (15) through (20) do not include the case of $\alpha = 0$, for which equation (12) cannot be treated as a Bessel equation. For completeness, the solution corresponding to $\alpha = 0$ needs to be investigated. Putting $\alpha = 0$ in equations (12) and (13) yields

$$R(r) = A_0 r+B_0\frac{1}{r}, \qquad Z(z) = C_0\,e^{\lambda'_0 z}+D_0\,e^{\lambda''_0 z} \tag{21}$$

where

$$\lambda'_0 = \tfrac{1}{2}[k+\sqrt{k^2-4\beta^2(\tilde{C}_{66}/\tilde{C}_{44})}],$$
$$\lambda''_0 = \tfrac{1}{2}[k-\sqrt{k^2-4\beta^2(\tilde{C}_{66}/\tilde{C}_{44})}], \tag{22}$$

and A_0, B_0, C_0 and D_0 are the integration constants.

In this case, the expression for the displacement is

$$v_2(r, z, t) = \left(A_0 r + B_0 \frac{1}{r}\right)(C_0\, e^{\lambda\delta z} + D_0\, e^{\lambda\delta' z})\, e^{i\omega t}, \tag{23}$$

and the corresponding stress components are

$$\tau''_{\theta z} = \tilde{C}_{44}\left\{A_0 r + \frac{B_0}{r}\right\}\{C_0 \lambda'_0\, e^{(\lambda\delta - k)z} + D_0 \lambda''_0\, e^{(\lambda\delta' - k)z}\}\, e^{i\omega t}, \tag{24}$$

and

$$\tau''_{r\theta} = -\tilde{C}_{66}\frac{2B_0}{r^2}\{C_0\, e^{(\lambda\delta - k)z} + D_0\, e^{(\lambda\delta' - k)z}\}\, e^{i\omega t}. \tag{25}$$

The inner lateral surface, $r = a$, and the outer lateral surface, $r = b$, of the tube are free from tractions. The boundary conditions on the surfaces are

$$\tau_{r\theta} = 0 \quad \text{at} \quad r = a, b. \tag{26}$$

Substitution of equation (20) into equation (26) yields the characteristic equation

$$J_2(\alpha a)Y_2(\alpha b) - J_2(\alpha b)Y_2(\alpha a) = 0, \tag{27}$$

where the α must be a root of equation (27). Let the set of roots be $\alpha_i (i = 1, 2, \ldots)$ which are known to be infinite in number, all simple and real [8]. Only the first four roots are calculated for various ratios of radii $\phi = a/b$, and listed in Table 1. For each characteristic value α_j, the relation existing between the integration constants A_j and B_j is

$$B_j = H_j A_j, \tag{28}$$

where

$$H_j = -\frac{J_2(\alpha_j a)}{Y_2(\alpha_j a)} = -\frac{J_2(\alpha_j b)}{Y_2(\alpha_j b)}.$$

In the same manner, from the boundary conditions (26), equation (25) yields

$$B_0 \equiv 0. \tag{29}$$

TABLE 1. FIRST FOUR ROOTS OF THE CHARACTERISTIC EQUATION

$\left(\text{Let } \Sigma = ab, \phi = \frac{a}{b}\right)$

ϕ	Σ_1	Σ_2	Σ_3	Σ_4
0·00	5·136	8·417	11·620	14·796
0·05	5·136	8·420	11·630	14·821
0·10	5·142	8·457	11·739	15·444
0·20	5·222	8·804	12·494	16·190
0·30	5·470	9·600	13·905	18·290
0·40	5·966	10·894	15·997	21·164
0·50	6·814	12·855	19·046	25·281
0·60	8·227	15·904	23·693	31·515
0·70	10·720	21·070	31·501	41·950

Finally, the general expression for the displacement is

$$v(r,z,t) = v_1 + v_2 = \left\{ \sum_{i=1}^{\infty} C_i F_1(\alpha_i r)\, e^{\lambda_i' z} + \sum_{i=1}^{\infty} D_i F_1(\alpha_i r)\, e^{\lambda_i'' z} + C_0 r\, e^{\lambda_0' z} + D_0 r\, e^{\lambda_0'' z} \right\} e^{i\omega t}, \tag{30}$$

where the function $F_1(\alpha_i r)$ is defined as

$$F_1(\alpha_i r) = J_1(\alpha_i r) + H_i Y_1(\alpha_i r). \tag{31}$$

The non-vanishing stress components can be expressed as

$$\begin{aligned} \tau_{\theta z} = \tilde{C}_{44} \Bigg\{ & \sum_{i=1}^{\infty} C_i \lambda_i' F_1(\alpha_i r)\, e^{(\lambda_i' - k)z} + C_0 \lambda_0' r\, e^{(\lambda_0' - k)z} \\ & + \sum_{i=1}^{\infty} D_i \lambda_i'' F_1(\alpha_i r)\, e^{(\lambda_i'' - k)z} + D_0 \lambda_0'' r\, e^{(\lambda_0'' - k)z} \Bigg\} e^{i\omega t}, \end{aligned} \tag{32}$$

$$\tau_{r\theta} = -\tilde{C}_{66} \left\{ \sum_{i=1}^{\infty} C_i \alpha_i F_2(\alpha_i r)\, e^{(\lambda_i' - k)z} + \sum_{i=1}^{\infty} D_i \alpha_i F_2(\alpha_i r)\, e^{(\lambda_i'' - k)z} \right\} e^{i\omega t}, \tag{33}$$

where the function $F_2(\alpha_i r)$ is defined as

$$F_2(\alpha_i r) = J_2(\alpha_i r) + H_i Y_2(\alpha_i r). \tag{34}$$

Let $\{F_1(\alpha_i r)\}$ be defined as a set of the functions

$$F_1(\alpha_1 r), F_1(\alpha_2 r), \ldots, F_1(\alpha_n r), \ldots,$$

and

$$G_i = \tfrac{1}{2}\{b^2[F_1(\alpha_i b)]^2 - a^2[F_1(\alpha_i a)]^2\}, \qquad (i = 1, 2, \ldots). \tag{35}$$

It is shown in the appendix that $F_1(\alpha_n r)$ form a set of continuous orthogonal functions on a closed domain $[a, b]$ related to the positive integrable weight function r; i.e.

$$\int_b^a r F_1(\alpha_n r) F_1(\alpha_m r)\, dr = G_n \delta_{mn} \tag{36}$$

where δ_{mn} denotes the Kronecker delta defined by

$$\delta_{mn} = \begin{cases} 1 & (m = n), \\ 0 & (m \neq n). \end{cases}$$

Also note that from the recurrence formulas [9],

$$\frac{d}{dz}\{z^n J_n(z)\} = z^n J_{n-1}(z)$$

and

$$\frac{d}{dz}\{z^n Y_n(z)\} = z^n Y_{n-1}(z),$$

and equation (27), the following useful formula is obtained:

$$\int_b^a r^2 F_1(\alpha_i r)\, dr = 0. \tag{37}$$

The constants C_0, D_0, C_i and D_i $(i = 1, 2, \ldots)$ in equation (30) are determined by the boundary conditions at the ends of the cylinder. Assume that the shearing stresses are specified at both ends as

$$\begin{aligned} \tau_{\theta z}(r, 0, t) &= \tau_0(r)\,\mathrm{e}^{i\omega t}, \\ \tau_{\theta z}(r, L, t) &= \tau_L(r)\,\mathrm{e}^{i\omega t}, \end{aligned} \tag{38}$$

where $\tau_0(r)$ and $\tau_L(r)$ are prescribed functions.

By the substitution of equation (32) in equations (38) and the use of equations (36) and (37), the integration ccnstants are readily obtained:

$$\begin{aligned} C_0 &= \frac{4\left(\mathrm{e}^{\lambda_0'' L}\int_a^b r^2\tau_0(r)\,\mathrm{d}r - \mathrm{e}^{kL}\int_a^b r^2\tau_L(r)\,\mathrm{d}r\right)}{\tilde{C}_{44}\lambda_0'(\mathrm{e}^{\lambda_0'' L} - \mathrm{e}^{\lambda_0' L})(b^4 - a^4)}, \\ D_0 &= \frac{4\left(\mathrm{e}^{kL}\int_a^b r^2\tau_L(r)\,\mathrm{d}r - \mathrm{e}^{\lambda_0' L}\int_a^b r^2\tau_0(r)\,\mathrm{d}r\right)}{\tilde{C}_{44}\lambda_0''(\mathrm{e}^{\lambda_0'' L} - \mathrm{e}^{\lambda_0' L})(b^4 - a^4)}, \\ C_i &= \frac{\mathrm{e}^{\lambda_i'' L}\int_a^b r\tau_0(r)F_1(\alpha_i r)\,\mathrm{d}r - \mathrm{e}^{kL}\int_a^b r\tau_L(r)F_1(\alpha_i r)\,\mathrm{d}r}{\tilde{C}_{44}\lambda_i'(\mathrm{e}^{\lambda_i'' L} - \mathrm{e}^{\lambda_i' L})G_i} \\ D_i &= \frac{\mathrm{e}^{kL}\int_a^b r\tau_L(r)F_1(\alpha_i r)\,\mathrm{d}r - \mathrm{e}^{\lambda_i' L}\int_a^b r\tau_0(r)F_1(\alpha_i r)\,\mathrm{d}r}{\tilde{C}_{44}\lambda_i''(\mathrm{e}^{\lambda_i'' L} - \mathrm{e}^{\lambda_i' L})G_i}, \qquad (i = 1, 2, \ldots). \end{aligned} \tag{39}$$

If, instead of the stresses, the components of displacement are specified at both ends of the tube, then, for example,

$$\begin{aligned} v(r, 0, t) &= 0 \\ v(r, L, t) &= V_L(r)\,\mathrm{e}^{i\omega t}. \end{aligned} \tag{40}$$

Substitution of equation (30) into equation (40) yields

$$\begin{aligned} C_i = -D_i &= \frac{\int_a^b rV_L(r)F_1(\alpha_i r)\,\mathrm{d}r}{G_i(\mathrm{e}^{\lambda_i' L} - \mathrm{e}^{\lambda_i'' L})} \qquad (i = 1, 2, \ldots), \\ C_0 = -D_0 &= \frac{4\int_a^b r^2 V_L(r)\,\mathrm{d}r}{(b^4 - a^4)(\mathrm{e}^{\lambda_0' L} - \mathrm{e}^{\lambda_0'' L})}. \end{aligned} \tag{41}$$

FREE TORSIONAL VIBRATION

The case of free torsional vibration of a tube can be considered as a special case of the previous study with the rigid mode $(\alpha = 0)$ and the modal mode $(\alpha \neq 0)$ considered separately. All the boundary surfaces of the tube must be free from tractions; i.e.

$$\tau_{\theta z}(r, 0, t) = \tau_{\theta z}(r, L, t) = \tau_{r\theta}(a, z, t) = \tau_{r\theta}(b, z, t) = 0. \tag{42}$$

Thus the functions τ_0 and τ_L defined in equations (38) must vanish everywhere.

(i) *Rigid mode* $(\alpha = 0)$

Consider the rigid mode of the free torsional vibration of tubes. From equations (39), it is readily seen that the constants C_0 and D_0 cannot be determined, and that the constants

$C_i = D_i = 0, (i = 1, 2, \ldots)$. Also, in order to have all surfaces of the tube free from tractions, the following relation must be satisfied:

$$e^{\lambda_0'' L} - e^{\lambda_0' L} = 0. \tag{43}$$

It is clear that the values of λ_0' and λ_0'' must be complex conjugates, for otherwise equation (43) yields $\lambda_0' = \lambda_0''$, and the solutions obtained in previous sections are not correct. Thus, let δ be $\sqrt{[4\beta^2(\tilde{C}_{66}/\tilde{C}_{44}) - k^2]}$; then equation (22) can be rewritten as

$$\lambda_0' = \tfrac{1}{2}k + i\delta, \qquad \lambda_0'' = \tfrac{1}{2}k - i\delta. \tag{44}$$

Equation (43) results in

$$\delta = \frac{n\pi}{L}, \qquad (n = \pm 1, \pm 2, \ldots).$$

The natural frequencies of rigid modes can be obtained as

$$\omega_{0,n} = \frac{\tilde{C}_{44}}{\sqrt{(\tilde{\rho}\tilde{C}_{66})}}\sqrt{\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{k}{2}\right)^2\right]}, \tag{45}$$

and the corresponding modal shape is

$$v = E_n r\, e^{kz/2}\left(k \sin\frac{n\pi z}{L} - 2\frac{n\pi}{L}\cos\frac{n\pi z}{L}\right) e^{i\omega t}, \tag{46}$$

where E_n denotes the amplitude of oscillation.

For a homogeneous isotropic circular tube, i.e. for $k = 0$ and $\tilde{C}_{44} = \tilde{C}_{66} = G$, the shear modulus, equations (45) and (46) yield

$$\omega_{0,n} = \sqrt{\left(\frac{G}{\rho}\right)}\left(\frac{n\pi}{L}\right) \quad \text{and} \quad \frac{v}{r} = E_n \cos\frac{n\pi z}{L} e^{i\omega t},$$

as obtained by Love [1].

(ii) *Nodal mode* ($\alpha \neq 0$)

For this special mode of motion, in the circular tube there exists not only the component τ_{rz} but also the component $\tau_{t\theta}$ of shearing stress. Moreover, the modes of torsional vibration are more complex than the ones of the previous case. Now from equation (39) it may be readily seen that $C_0 = D_0 = 0$ and that the constants C_i and D_i (i = 1, 2, ...) cannot be determined. In order to satisfy the boundary conditions (42) it is required that

$$e^{\lambda_i' L} - e^{\lambda_i'' L} = 0 \qquad (i = 1, 2, \ldots), \tag{47}$$

where λ_i' and λ_i'' are defined in equation (17) except for the subscript. The subscript i denotes that the values of λ' and λ'' correspond to the characteristic value α_i which is a root of equation (27). As argued in the previous case, it can be concluded that λ_i' and λ_i'' must be complex conjugates. Therefore, the natural frequencies associated with the nodal mode of torsional vibration of tubes are

$$\omega_{i,m} = \sqrt{\left\{\frac{(\tilde{C}_{44})^2}{\tilde{\rho}\tilde{C}_{66}}\left(\frac{\tilde{C}_{66}}{\tilde{C}_{44}}\alpha_i^2 + \frac{k^2}{4} + \frac{m^2\pi^2}{L^2}\right)\right\}}, \qquad m = \pm 1, \pm 2, \ldots \tag{48}$$

where α_i $(i = 1, 2, \ldots)$ are the roots of equation (27).

The corresponding modal shapes and stress components are

$$v_{i,m} = E_{i,m}\, e^{kz/2}\left\{k \sin\frac{m\pi z}{L} - 2\frac{m\pi}{L}\cos\frac{m\pi z}{L}\right\} F_1(\alpha_i r)\, e^{i\omega t},$$

$$(\tau_{\theta z})_{i,m} = E_{i,m}\tilde{C}_{44}\left\{J_1(\alpha_i r) - \frac{J_2(\alpha_i a)}{Y_2(\alpha_i a)} Y_1(\alpha_i r)\right\} \times \left\{e^{-kz/2}\left(\frac{k^2}{2} + 2\frac{m^2\pi^2}{L^2}\right)\sin\frac{m\pi z}{L}\right\} e^{i\omega t}, \tag{49}$$

$$(\tau_{r\theta})_{i,m} = -\tilde{C}_{66}E_{i,m}\, e^{-kz/2}\left\{k \sin\frac{m\pi z}{L} - 2\frac{m\pi}{L}\cos\frac{m\pi z}{L}\right\} \times e^{i\omega t}\alpha_i\left\{J_2(\alpha_i r) - \frac{J_2(\alpha_i a)}{Y_2(\alpha_i a)} Y_2(\alpha_i r)\right\}.$$

The equations of natural frequency (45) and (46) are rewritten in non-dimensional forms as

$$\frac{\omega_{0,n}}{(\tilde{C}_{44}/\tilde{\rho}b^2)^{\frac{1}{2}}} = \sqrt{\{[(2n\pi)^2 + \phi_k^2]/4\phi_c\phi_e^2\}}, \tag{50}$$

$$\frac{\omega_{i,m}}{(\tilde{C}_{44}/\tilde{\rho}b^2)^{\frac{1}{2}}} = \sqrt{\{\Sigma_i^2 + [(2m\pi)^2 + \phi_k^2]/4\phi_c\phi_e^2\}}, \tag{51}$$

where $\phi_k = kL$, $\phi_c = \tilde{C}_{66}/\tilde{C}_{44}$, $\phi_e = L/b$, $\phi = a/b$, and $\Sigma_i = \alpha_i b$ are all non-dimensional parameters.

The ϕ_k characterizes the homogeneity of the material of the circular tube. In practice, $-0{\cdot}5 \le \phi_k \le +0{\cdot}5$. The parameter ϕ_c describes the anisotropy of the medium. For the case of isotropic media the parameter ϕ_c equals unity. According to the data given by Mason [10], $0{\cdot}4 \le \phi_c \le 2{\cdot}6$. The ratio of radii ϕ is assumed to be $0 \le \phi \le 0{\cdot}7$. When $\phi = 0{\cdot}0$ the body is a solid cylinder. The range of the length parameter ϕ_e considered is $0{\cdot}4 \le \phi_e \le 2{\cdot}5$.

The numerical values of natural frequencies were calculated using an IBM 360 digital computer. Figure 1 shows the effects of anisotropy at the lowest natural frequency of the nodal modes $\omega_{1,1}$ with $\phi_k = 0$ and $\phi_e = 1{\cdot}0$, for various values of ϕ. It is clear that the influences of anisotropy of material on natural frequencies of torsional vibration are very significant. The effects of the ratio of radii ϕ at $\omega_{0,1}$ and $\omega_{1,1}$, the lowest natural frequencies of torsional vibration for the rigid mode and for the nodal mode respectively, are shown in Fig. 2. Note particularly that ϕ does not affect the lowest natural frequency of the rigid mode $\omega_{0,1}$. Therefore, the frequencies of this particular mode are independent of the inner radius, even if the inner radius is zero. However, the natural frequencies of the nodal mode $\omega_{i,m}$ are changed appreciably by the change of ϕ values. The effects of the length parameter ϕ_e on $\omega_{1,1}$ are shown in Fig. 3. For circular rings ($0{\cdot}4 \le \phi_e \le 1{\cdot}0$) the influences ϕ_e on the natural frequency $\omega_{1,1}$ are quite significant, while for long circular tubes ($\phi_e > 2{\cdot}0$) the effects are insignificant. Since the value of ϕ_k^2 is very small in comparison with other terms inside the square root of equations (50) and (51), the negligence of ϕ_k^2 does not change the natural of the equations. Thus, the effects of the parameter of homogeneity ϕ_k on the natural frequency are very small and negligible.

FORCED VIBRATION

Consider a circular tube with the end $z = 0$ fixed and the other end $z = L$ subjected either to a displacement or to an appropriate stress distribution.

If

$$v(r, 0, t) = 0$$

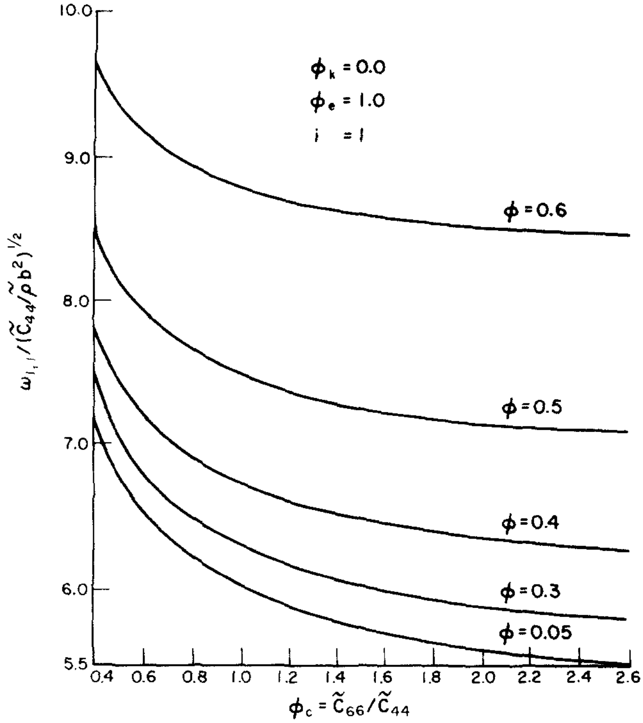

FIG. 1. The influences of anisotropy of material at the natural frequency.

and

$$\tau_{z\theta}(r, L, t) = K_1 r\, e^{i\omega t}, \tag{52}$$

then the substitution of equations (30) and (32) into equation (52) yields

$$C_i = -D_i = 0, \qquad (i = 1, 2, \ldots),$$

$$C_0 = -D_0 = \frac{K_1\, e^{kL}}{\tilde{C}_{44}[\lambda_0'\, e^{\lambda_0' L} - \lambda_0''\, e^{\lambda_0'' L}]}. \tag{53}$$

It is readily determined from equations (30), (32) and (33) that

$$v(r, z, t) = \frac{K_1\, e^{kL} r[e^{\lambda_0' z} - e^{\lambda_0'' z}]\, e^{i\omega t}}{\tilde{C}_{44}[\lambda_0'\, e^{\lambda_0' L} - \lambda_0''\, e^{\lambda_0'' L}]},$$

$$\tau_{\theta z} = \frac{K_1 r[\lambda_0'\, e^{\lambda_0' z} - \lambda_0''\, e^{\lambda_0'' z}]\, e^{i\omega t}}{\lambda_0'\, e^{\lambda_0' L} - \lambda_0''\, e^{\lambda_0'' L}}, \tag{54}$$

$$\tau_{r\theta} = 0.$$

The motion of this particular mode is associated with the motion wherein each transverse section of the cylinder rotates rigidly. This fact may be seen clearly from equation (54). The natural frequencies associated with this kind of mode of motion are given by equation (50). As the frequency of the driving force ω approaches one of the values $\omega_{0,n}$ given by equation (44), the amplitude of oscillation K_1 grows without limit and a resonance occurs.

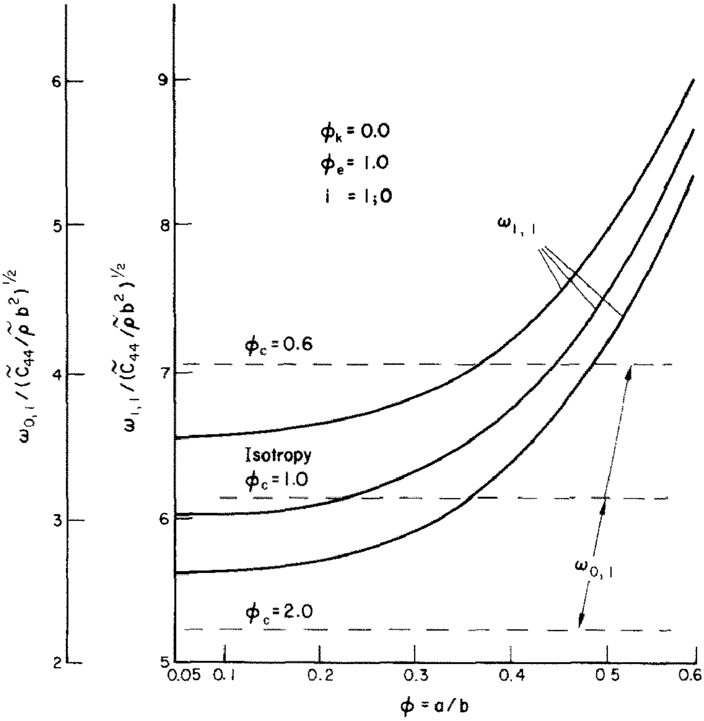

FIG. 2. The effects of the ratio of radii at the natural frequencies.

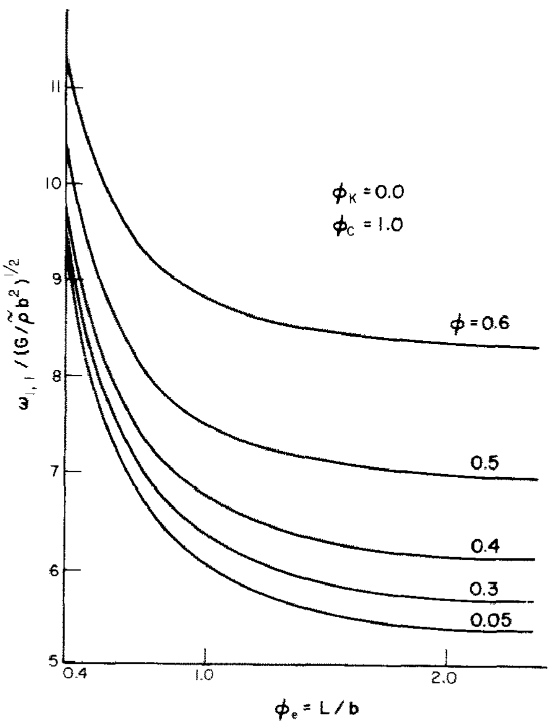

FIG. 3. The effects of the length parameter at natural frequency

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APPENDIX

Orthogonality of the eigenfunctions

Suppose the differential operator of equation (12),

$$\mathbf{T} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) + \left(\alpha^2 - \frac{1}{r^2}\right), \tag{A-1}$$

is to be defined on the domain $[a, b]$ by the boundary conditions (25),

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r}\right)u(r)\bigg|_{\substack{r=a\\r=b}} = 0, \tag{A-2}$$

where u is an element in the domain of $\mathbf{T}$. Assume u and v are the elements in the domain of $\mathbf{T}$; by definition the scalar product of two functions with respect to the positive weight function r is

$$\langle v, \mathbf{T}u\rangle = \int_a^b rv(\mathbf{T}u)\,\mathrm{d}r. \tag{A-3}$$

With integration by parts and the use of conditions (A-2), the scalar product (A-3) yields

$$\langle v, \mathbf{T}u\rangle = u(b)\left[v - \frac{\mathrm{d}v}{\mathrm{d}r}\right]_{r=b} - u(a)\left[v - \frac{\mathrm{d}v}{\mathrm{d}r}\right]_{r=a} + \langle \mathbf{T}v, u\rangle. \tag{A-4}$$

Then

$$\langle v, \mathbf{T}u\rangle = \langle \mathbf{T}v, u\rangle \tag{A-5}$$

where v satisfies the conditions

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r}\right)v(r)\bigg|_{\substack{r=a\\r=b}} = 0. \tag{A-6}$$

These boundary conditions are equivalent to (A-2). Thus the operator $\mathbf{T}$ defined by equations (A-1) and (A-2) is a self-adjoint operator with self-adjoint boundary conditions.

THEOREM. If T is a self-adjoint operator with self-adjoint boundary conditions, the eigenfunctions of T form an orthogonal set.

PROOF. Let u_j and u_k be two arbitrary distinct eigenfunctions of the differential operator $\mathbf{T}$, and λ_j and λ_k be the corresponding eigenvalues. Thus

$$\mathbf{T}u_j = \lambda_j u_j \quad \text{and} \quad \mathbf{T}u_k = \lambda_k u_k. \tag{A-7}$$

From equation (A-5)

$$\begin{aligned} 0 &= \langle u_j, \mathbf{T}u_k\rangle - \langle \mathbf{T}u_j, u_k\rangle \\ &= (\lambda_k - \lambda_j)\langle u_j, u_k\rangle. \end{aligned} \tag{A-8}$$

If $\lambda_k \neq \lambda_j$, equation (A-8) implies that $\langle u_j, u_k\rangle = 0$. Then the theorem is proven.

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Абстракт—Работа касается крутильных колебаний в конечной круглой трубе из анизотропного неоднородного материала. Получается общее решение задачи. Так свободные как и вынужденные крутильные колебания труб можно рассматривать в качестве специальных случаев задачи. Определяются значения свободных частот заключающих виды колебаний высших рядов для таких геометрических параметров как отношение радиуса трубы и ее длины. Решаются, также, значения свободных частот в выражениях для разных значений параметров анизотропии. Интересно, что наличие центрального отверстия в трубе не вызывает никогого эффекта на свободные частоты жесткого режима. В случае узловых видов колебаний, изменяются свободные частоты с толщиной трубы.